5. Summary and Conclusions

Equations have been presented for a gradient search algorithm to design feedback compensation transfer functions for general linear time-invariant systems. The organization of these equations to produce a computer program for automatic frequency-domain synthesis was discussed.

It was shown that a suitable method for avoiding convergence difficulties in minimizing the particular cost function treated involves the use of percentage changes in the transfer function coefficients.

A design problem involving the stabilization of a complex ballistic missile control system was chosen to demonstrate the performance of the design algorithm implemented in the computer program AUTO.

Practical experience indicates that for the ratio of polynomials in a complex variable, a cost function expressed in polar coordinates is preferable to its equivalent in rectangular coordinates. The polar formulation yields faster convergence and, in some cases, can be shown mathematically to eliminate local minima, which might prevent convergence to the true minimum where a rectangular cost function is utilized. These observations should also stimulate further research into the nature of such cost functions.

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OCTOBER 1970 AIAA JOURNAL VOL. 8, NO. 10

Structural Optimization in the Dynamics Response Regime: A Computational Approach

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A structural optimization problem is considered in which the design requirements include restrictions on the dynamic response and frequency characteristics of the structure. One of the central concerns of this phase of the work has been to overcome the problems inherent in treating systems with many degrees of freedom. The simple planar truss-frame with both distributed and concentrated mass is the model upon which this exploratory study is based. Limitations have been imposed upon maximum dynamic stresses and displacements (handled by the shock spectral approach) as well as on the natural frequencies of the structure. A "direct" optimization method (the method of feasible directions) which consists of a design-analysis cycle was used. Computationally efficient schemes are given for the necessary derivatives of maximum response and natural frequency. Numerical examples are given and computational effectiveness is indicated.

1. Introduction

THE importance of dynamic response in current structural problems has prompted an increased attention to the synthesis (automated optimum design) of structures for which the dynamic response will be a controlling criterion. This paper reports upon work that was undertaken to demonstrate the feasibility of using a dynamics technology within the structural synthesis framework. The design restrictions imposed include limitations on the dynamic displacements, stresses, and ranges in which the natural frequencies of the structure are allowed to fall.

As a first phase, the automated minimum weight design of the general planar truss-frame system‡ has been undertaken. The dynamic response is assumed to be linear, undamped, and the result of known externally applied exciting forces at the joints or of foundation displacements.

The structural design problems that arise within several of the more common design philosophies have been put in the form of mathematical programing problems. In the present work the minimum weight design of general planar truss-frame systems, subject to dynamic response limitations, has been cast as a mathematical programing problem. The resulting mathematical programing problem has been solved using the method of feasible directions.[1]

Presented at the AIAA Structural Dynamics and Aeroelasticity Specialist Conference, New Orleans, La., April 16–17, 1969 (no paper number; published in bound volume of conference papers); submitted April 29, 1969; revision received April 3, 1970. This work was supported in part by the National Aeronautics and Space Administration under research grant NsG 110-61.

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‡ The nomenclature "truss-frame" as used in this paper denotes a structure consisting of elements capable of axial and lateral deformations. Such elements may be rigidly connected or pinned together.

This application requires a number of capabilities to be assembled. First the class of structures has to be adequately represented by a model for dynamic analysis with an eye to the final embedment of its analysis in the iterative design-analysis loop of the optimization procedure. Likewise, the analysis of the model, that is, the numerical determination of the pertinent dynamic characteristics of the model, has to be performed in a way which is both consistent with and takes advantage of the iterative design cycle. And finally, for the solution of the mathematical programing problem, the rates of change (i.e., the partial derivatives) of the dynamic behavior of the model with respect to the design variables has to be determined.

In this paper each of these topics will be discussed in more or less detail depending upon the degree to which it has previously been covered in the open literature. The material is also covered in complete form in the Ph.D. thesis of Kapoor.[2] Several numerical examples are presented which give support to the effectiveness of the approach.

2. Mathematical Programing Problem

A classical mathematical programing problem is one in which a multivariable function $F(\mathbf{D})$ (where $\mathbf{D}$ is a n dimensional vector consisting of d_j, $j = 1, 2, \ldots, n$) is to be minimized subject to given constraints $g_i(\mathbf{D}) \leq 0$, $i = 1, 2, \ldots, m$. The function $F(\mathbf{D})$ is called the objective function and its choice is governed by the nature of the problem.

In much of structural design it is taken as axiomatic that the objective is to use a minimum amount of material or, almost equivalently, to design a minimum weight structure. The issue is debatable, of course, since cost or other performance characteristics are often of overriding concern. Nonetheless, weight minimization appears to continue to be of sufficient interest to warrant its use as the objective function in structural development.

The present work concerns the minimum weight design of a structure composed of straight tubular members, hinged or rigidly connected at their ends, and all lying in a single plane. The structure may, in addition to its distributed mass, have a number of concentrated mass elements placed at some joints. It is assumed that the base or ground is given a shock motion and that this motion gives rise to dynamic displacements and distortions within the structure which in turn cause time dependent stresses within the members. Furthermore, it is assumed that there are some steady harmonic forcing components of moderate amplitude in the environment which, at resonance, could cause damaging motion but which are deemed to be unimportant if they fall outside of the resonance bands of the structure.

The design criteria are taken to be 1) that the relative displacements due to the shock must not exceed certain prescribed limits, 2) the stresses in the members must not exceed certain limits, and 3) the natural frequencies of the structure are excluded from certain bands. These criteria are simplified rules prescribed to avoid failure of the structure due to the shock input and the mild harmonic forcing. They ignore the static loadings, should they exist, fatigue, dynamic buckling and other possibly important considerations. These simplifications have been made, first because even though they are not fully cognizant of the practical situation, they still represent some realistic modes of failure and, secondly, because for the feasibility objective of the work, they are sufficiently representative of the nature of real restrictions.

The design criteria and the objective function being known, the structural optimization problem for the truss-frame structure can be cast as a mathematical programing problem once we decide the shape of members and choose the design variables. For the purpose of the present work tubular members of uniform cross section were used. This choice is one of the simplest and exhibits characteristics similar to those of other special shapes used in practical problems. In order to further simplify the problem, only the mean diameters of the tubular members were taken as design variables and the wall thickness kept constant. The choice of the mean diameters of the members as the design variables gives one variable per member and it is a parameter which governs both the area of cross section of the member and its moment of inertia. Very often in other structural problems it is possible to perform similar simplifications, obtaining a single parameter by which the member can be completely characterized. This point, however, is not central to the present application.

The optimization problem can be expressed as
Minimize

$$W(\mathbf{D}) = \sum_{j=1}^{m} \pi d_j t_j l_j \rho_j \tag{1}$$

subject to

$$|Y(\mathbf{D},\mathbf{V},t)| - Y^{(u)} \leq 0,\ t > 0 \tag{2}$$

for all points in the structure (i.e., for all $\mathbf{V}$)

$$|\sigma(\mathbf{D},\mathbf{V},t)| - \sigma^{(u)} \leq 0,\ t > 0 \tag{3}$$

for all points in the structure (i.e., for all $\mathbf{V}$)

$$\lambda_i^{(l)} \leq \lambda_i \leq \lambda_i^{(u)} \qquad i = 1, \ldots, p \tag{4}$$

and

$$d_j^{(l)} \leq d_j \leq d_j^{(u)} \qquad j = 1, \ldots, m \tag{5}$$

where W represents the weight of the structure, $\mathbf{D}$ is the vector of design variables, d_j is the mean diameter of the tubular member j of the truss-frame structure, t_j is the wall thickness of the jth member, l_j is the length of the jth member, ρ_j is the weight density of the jth member, $Y(\mathbf{D},\mathbf{V},t)$ represents the displacement at any point $\mathbf{V}$ on the structure as a function of time t, $Y^{(u)}$ is the upper limit on the displacement, $\sigma(\mathbf{D},\mathbf{V},t)$ represents the stress at any point $\mathbf{V}$ on the structure as a function of time t, $\sigma^{(u)}$ is the upper limit on the stress, perhaps taken to be the yield point of the material. (Note that in this present work, the material has been considered to have the same limit stress in tension as in compression), λ_i represents the ith eigenvalue of the structure (eigenvalues are the square of natural frequencies), $\lambda_i^{(u)}$ is the upper limit on the ith eigenvalue $\lambda_i^{(l)}$ is the lower limit on the ith eigenvalue, p is the number of natural frequencies, starting from the lowest end of the spectrum, which are constrained, $d_j^{(u)}$ is the upper limit on the mean diameter of the jth member, $d_j^{(l)}$ is the lower limit on the mean diameter of the jth member, m is the number of members.

The "side constraints," Eq. (5), impose the limits on the size of the design variables.

The objective function, Eq. (1), is a linear function of the design variables and the side constraints, Eq. (5), are also linear inequalities. However, the behavior constraints, Eqs. (2–4) are, in general, nonlinear and hence the mathematical programing problem formulated previously is a nonlinear programing problem. Moreover, in the form given above the constraints Eqs. (2) and (3) are "parametric," that is, they require satisfaction over a range of some parameter, in this case, the time parameter t and the space parameter $\mathbf{V}$, the location on the structure. These will be simplified at a later stage.

When the mean diameter of each tubular member is selected to be a design variable, then the design vector $\mathbf{D}$ consists of m components d_j, $j = 1, \ldots, m$. However, it is quite likely that several members of the structure are desired to be of the same diameter continuously in the design space. This imposes additional "linking" constraints on the optimization problem which can be handled rather easily.[2]

3. Structural Model

The general planar truss-frame structure is idealized as a system of straight members lying in a plane and interconnected at joints. The members have their axes of symmetry in the plane and the translation of the members are in the plane of the structure. All applied forces are assumed to act in the plane of the structure. The static analysis of a structure having hinged joints when subjected only to joint loads results in axial forces of tension or compression in the members. In general, however, dynamic loads result in bending couples, shearing forces and axial forces in the plane of the structure at any cross section of its members. A simple study supporting the necessity to go to this slightly more complex truss-frame model for dynamic response is given in Ref. 2.

The discrete element method of structural analysis is used in the present work. The matrix formulation of the general structural dynamic response problem, in the absence of damping, results in the equation

$$[M]\ddot{\mathbf{Y}} + [K]\mathbf{Y} = \mathbf{F}(t) \tag{6}$$

where $[M]$ and $[K]$ are, respectively, the master mass and the master stiffness matrix of the structure and their order n corresponds to the elastic degrees of freedom of the system. The vectors, $\mathbf{Y}$, $\ddot{\mathbf{Y}}$, and $\mathbf{F}(t)$ represent the displacement, acceleration and effective load, respectively. Ordinarily $[K]$ and $[M]$ are assembled from the corresponding element matrices and the various steps in the analysis of the model are carried out by operating upon these matrices. In the analysis methods used here, the assembly of $[K]$ and $[M]$ was avoided as will be discussed in the next section.

Essentially the element matrices are derived from the standard beam and truss elements (see for example Ref. 3) wherein the axial deformation is assumed to be a linear function of the distance along the member and the transverse deformation taken as a cubic function of the distance. This gives the element six degrees of freedom: two displacements and a rotation at each end. The so-called consistent mass matrix is used.[4]

In order to represent adequately the dynamic behavior of some of the structures with long members, these were modeled with several elements and thus there is a distinction between "members" and "elements."

4. Analysis of the Structural Model

There are two distinct phases to the dynamic analysis as it is performed in this application. First an eigenvalue analysis of the structural model is performed and second the dynamic response is characterized. The familiar approach, to recall it briefly, is to solve the eigenproblem

$$[K]\mathbf{X} = \lambda[M]\mathbf{X} \tag{7}$$

for the λ_i, $\mathbf{X}_i$ pairs and make the transformation or change of variables

$$\mathbf{Y} \simeq \sum_{i=1}^{r} q_i\mathbf{X}_i \equiv [Q]\mathbf{q} \tag{8}$$

where

$$[Q] \equiv [\mathbf{X}_1,\mathbf{X}_2 \ldots \mathbf{X}_r] \tag{9}$$

and where r in many practical situations is taken to be considerably less than n as an approximation to save computational effort. Thus the original system Eq. (6) can be written, after premultiplication by $[Q]^T$ as

$$[Q]^T[M][Q]\ddot{\mathbf{q}} + [Q]^T[K][Q]\mathbf{q} = [Q]^T\mathbf{F}(t) \tag{10}$$

or, in view of the orthonormality of the eigenvectors

$$\ddot{\mathbf{q}} + [\Lambda]\mathbf{q} = [Q]^T\mathbf{F}(t) \tag{11}$$

where $[\Lambda]$ is the diagonal matrix with diagonal elements λ_i. Thus the original problem is reduced to the set of uncoupled equations.

$$\ddot{q}_i + \lambda_i q_i = \tilde{f}_i(t); \quad i = l, \ldots, r \tag{12}$$

In this application, we are considering the forcing function to be such that

$$\tilde{f}_i(t) = c_i f(t); \quad i = 1, \ldots, 2 \tag{13}$$

This situation commonly occurs when the structure is given a ground induced shock, because the right hand side of Eq. (6) becomes

$$\mathbf{F}(t) = -[M]\ddot{\mathbf{Z}}(t) \tag{14}$$

where $\ddot{\mathbf{Z}}(t)$ represents the acceleration of the base and now $\mathbf{Y}$ represents the relative displacement between the structure and the base in Eq. (6).

If $\mathbf{Z}(t)$ is further specialized to a unidirectional shock then we can write

$$\ddot{\mathbf{Z}}(t) = \hat{e}\ddot{\xi}_g(t) \tag{15}$$

where $\hat{e}$ is a unit vector giving the "direction" of the shock in generalized coordinates and $\ddot{\xi}_g(t)$ is a scalar magnitude of the base acceleration.

Substituting Eqs. (14) and (15) into Eq. (11) we finally obtain the equations of motion for the participation coefficients as

$$\ddot{q}_i + \lambda_i q_i = c_i\ddot{\xi}_g(t) = -\mathbf{X}_i{}^T[M]\hat{e}\ddot{\xi}_g(t) \tag{16}$$

Eigenproblem

It is assumed that in practical problems the order of $[K]$ and $[M]$ is so large that it is prohibitively expensive to obtain a complete eigensolution and further that even a partial solution by ordinary methods is too time consuming to perform repeatedly. A method developed to suit the special characteristics of the present problem is reported in Ref. (5). Briefly it consists of minimizing the Rayleigh quotient

$$R(\mathbf{X}) \equiv \mathbf{X}^T[K]\mathbf{X}/\mathbf{X}^T[M]\mathbf{X} \tag{17}$$

in a sequence of subspaces for as many of the eigenvectors and eigenvalues as desired. Its principal advantages for the problem at hand are: 1) it does not require the assembly of $[K]$ and $[M]$ nor the use of any other large matrices and 2) it is iterative, converging quite rapidly given good initial starting points. This latter is particularly advantageous in an iterative design process since as the design is evolved the natural frequencies may change drastically but the mode shapes usually change only gradually. Thus the mode shapes for a previous design are good candidates for starting points for a new design. In the present application a partial eigensolution is used for an approximation to the dynamic response. A first design is analyzed using random vectors as starting iterates for the eigenvalue analysis and then for all subsequent analyses the most recent set of eigenvectors are utilized by checking each of them for suitability as a starting vector for each of the eigenvalues. This procedure usually makes it possible to obtain the new partial eigensolution in a small fraction of the time required for the original solution and for large systems it appears to be quite efficient over-all compared to standard methods.

Response Problem

In the optimization problem given by Eqs. (1–5) the stress and displacement constraints [Eqs. (2) and (3)] contain time and spatial parameters. A more common form of expressing such constraints is

$$\max_{t>0}\left\{\max_{\text{all } \mathbf{V}} \left[|Y(\mathbf{D},\mathbf{V},t)| - Y^{(u)}\right]\right\} \leq 0 \tag{18}$$

$$\max_{t>0} \left\{ \max_{\text{all } \mathbf{V}} \left[|\sigma(\mathbf{D},\mathbf{V},t)| - \sigma^{(u)} \right] \right\} \leq 0 \tag{19}$$

The parameter **V** is disposed of, for the case of displacements, simply by placing the restrictions only on the nodal displacements. Of course, it is possible that large displacements can occur along the member and if this is suspected then "virtual" nodes can be introduced at points along the members. Usually if a member is expected to have such behavior it would be modeled with two or more elements in any event to represent the response more accurately.

A similar situation exists with the stresses. If most of the stress is axial in a member then it is of little consequence at what point along the member the stress is evaluated, on the other hand if the member has significant bending then the location of the maximum stress becomes problematical. The approach used here is to compute the stress at each end of each *element* used to model the structure. It is a peculiarity of the element idealization used that it represents the bending stress as a linear function of the distance along the element.[3] This produces the somewhat disconcerting result that the maximum bending stress occurs at the ends of the element. Again, however, for members in which significant bending is expected, a two-element model produces much more accurate results and allows the maximum stress to occur near the midpoint of the member.

Since mathematical programing problems with a parameter are quite difficult, it is preferable to eliminate t from the constraints as well as **V**. This is done in the present work by computing conservative, but generally realistic upper bounds in time for the displacements and stresses at the nodes. An estimate of the upper bound is obtained by using the shock spectral approach as described below.

Given an equation of motion of the form of Eq. (12)

$$\ddot{\eta} + \lambda\eta = \ddot{\xi}_g(t) \tag{20}$$

where λ is the natural frequency squared, we can form the function

$$\eta_m(\lambda) = \max_{t>0}[|\eta(\lambda,t)|] \tag{21}$$

This is equivalent to considering a specific shock pulse [such that $\ddot{\xi}_g(t) = 0,\ t < 0,\ t > t_0$] and defining a function $\eta_m(\lambda)$ as the maximum of the response of a single degree-of-freedom oscillator. This function varies with the natural frequency of the oscillator. The function $\eta_m(\lambda)$ (or its plot) is called the shock spectrum of $\ddot{\xi}_g(t)$. Such spectra are available for a wide variety of known pulses $\ddot{\xi}_g(t)$ (see Ref. 6). To determine the maximum q_{im} of any of Eqs. (16) given the spectrum of $\ddot{\xi}_g(t)$ we compute simply

$$q_{im} = |c_i|\eta_m(\lambda_i) \tag{22}$$

The maxima of the modal responses do not all occur at the same time nor do they all have the same actual sign [note the absolute value signs in Eq. (22)]. However, an upper bound on displacement can be obtained from a superposition of the maximum modal responses. Using the transformation Eq. (8) with $r = n$ we have the absolute maximum value of any displacement $y_j(t)$ as

$$|y_j(t)| = \left|\sum_{i=1}^{n} q_i(t)\mu_{ji}\right| \leq \sum_{i=1}^{n} |q_i(t)\mu_{ji}| \leq \sum_{i=1}^{n} q_{im}|\mu_{ji}| \equiv y_{jm} \tag{23}$$

where μ_{ji} represents the jth component of the ith eigenvector $\mathbf{X}_i$ and y_{jm} represents the jth component of the upper-bound displacement vector $\mathbf{Y}_m$.

For a fairly reliable, but not perfectly rigorous, bound the summation is taken to $r < n$ as mentioned earlier. Thus the original constraint on displacements is replaced by one of the form

$$y_{jm} - Y^{(u)} \leq 0; \quad \text{for all } j \tag{24}$$

A similar but slightly more complex procedure is organized for the stresses at the selected points. Here the maximum stress in each mode for the particular site is calculated and then these are superposed in accordance with the weighting of the modal participation factors to produce a σ_{jm}. The constraints are then taken to be

$$\sigma_{jm} - \sigma^{(u)} \leq 0; \quad \text{for all } j \tag{25}$$

where σ_{jm} is a rigorous upper bound if all modes are used and almost invariably remains conservative if $r < n$ modes are used.

5. About the Method of Feasible Directions

The method of feasible directions is a direct optimization technique which uses the design cycle

$$\mathbf{D}_{q+1} = \mathbf{D}_q + \alpha\mathbf{S}_q \tag{26}$$

where $\mathbf{D}_{q+1}$ is a new design, $\mathbf{D}_q$ is the previous design, $\mathbf{S}_q$ is a "useable feasible" redesign vector, and α is always selected so that the design $\mathbf{D}_{q+1}$ satisfies the constraints $g_j(\mathbf{D}) \leq 0$. This is accomplished by determining a direction $\mathbf{S}_q$ such that

$$\mathbf{S}_q{}^T\nabla g_j \leq 0,\ j\epsilon J \tag{27}$$

where J represents the indices of the active constraints $[g_j(\mathbf{D}) = 0,\ j\epsilon J]$. Such a vector is called a *feasible* vector. A feasible vector which also satisfies

$$\mathbf{S}_q{}^T\nabla F \leq 0 \tag{28}$$

is called a *useable feasible* vector. Equations (27) insure that the $\mathbf{S}_q$ vector is directed into the feasible portion of the design space, at least for some finite distance, and Eq. (28) that F is decreasing in the direction $\mathbf{S}_q$. Vectors satisfying these inequalities can be found by a variety of means[1] and the method used in this work is based upon establishing a criterion for "optimizing" the direction and then solving a linear programing problem involving the inequalities Eqs. (27) and (28). The method is relatively effective on nonlinear programing problems of the type at hand; in particular it is especially well suited to reducing the number of new analyses required for performing the redesign cycles Eq. (26).

One important facet of the method is its requirement for the gradients of the active constraints in order to calculate the redesign vector. This means that ultimately derivatives with respect to the design variables of the quantities y_{jm} and σ_{jm} [see Eqs. (24) and (25)] must be taken. It is because this information is used, that the method is so powerful but on the other hand it is not immediately apparent how these quantities can be computed efficiently. This is discussed in the following section.

6. Derivatives of the Constraint Functions

The dynamic response quantities, as formulated previously depend explicitly upon the eigenvalues and associated eigenvectors. Furthermore, limitations on the eigenvalues themselves are also considered to be among the constraints. Thus in order to take derivatives of the constraint functions we must compute derivatives of the eigenvalues and eigenvectors. A simple method has been developed for these derivatives[7–9] and has been used in this work. The basic formulas are summarized below.

Derivatives of the Eigensolutions

For the derivative of an eigenvalue λ_i [Ref. (7)] with respect to a parameter of the matrices $[K]$ and $[M]$ (in this case a design variable) we get

$$\partial\lambda_i/\partial d_j = \mathbf{X}^T\{\partial[K]/\partial d_j - \lambda_i\partial[M]/\partial d_j\}\mathbf{X}_i \tag{29}$$

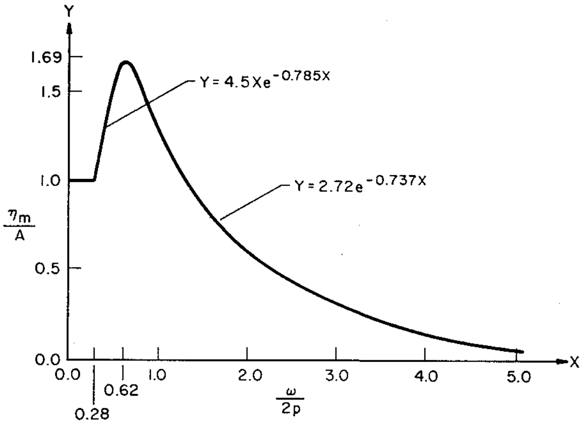

Fig. 1 Idealized shock spectrum for the one-half cycle sine shock pulse.

where $\mathbf{X}_i$ is assumed to be normalized so that

$$\mathbf{X}_i^T[M]\mathbf{X}_i = 1 \tag{30}$$

Notice that once a particular λ_i and $\mathbf{X}_i$ are determined these derivatives can easily be calculated because presumably the dependence of $[K]$ and $[M]$ upon the design variables are known.

The derivative of an eigenvector with respect to the design variables can also be expressed exactly, although not quite as simply. Since the eigenvectors of the present problem, in general, are linearly independent we can express any n-component vector as a linear combination of the $\mathbf{X}_k$, $k = 1, \ldots, n$. Thus

$$\partial\mathbf{X}_i/\partial d_j = \sum_{k=1}^{n} a_{ijk}\mathbf{X}_k \tag{31}$$

for which it has been shown in Ref. (7) that

$$a_{ijk} = \frac{\mathbf{X}_k^T\{\partial[K]/\partial d_j - \lambda_i\partial[M]/\partial d_j\}\mathbf{X}_i}{(\lambda_i - \lambda_k)}, \; k \neq i \tag{32}$$

and

$$a_{iji} = -\frac{\mathbf{X}_i^T\{\partial[M]/\partial d_j\}\mathbf{X}_i}{2} \tag{33}$$

where again the $\mathbf{X}_i$ are assumed to be $[M]$-orthonormal. A disadvantage of this formula is that, as Eq. (31) indicates, we would need the complete eigensolution to compute the needed derivatives. This is overcome by approximating $\partial\mathbf{X}_i/\partial d_j$ by a partial sum on k, say to $k = r$ instead of n. The approximation is to be consistent with that used in the dynamic analysis itself and it appears to have worked well in the present problem.

Derivatives of the Response Quantities

Turning back to Eq. (23) we see that the derivative of the upper bound displacement is

$$\frac{\partial y_{km}}{\partial d_j} = \sum_{i=1}^{r}\left\{\frac{\partial|c_i|}{\partial d_j}\,\eta_m(\lambda_i)|x_{ki}| + |c_i|\,\frac{\partial\eta_m(\lambda_i)}{\partial d_j}\,|x_{ki}| + |c_i|\eta_m(\lambda_i)\,\frac{\partial|x_{ki}|}{\partial d_j}\right\} \tag{34}$$

It should be noted that c_i depends (for ground induced shock motion) upon $[M]$ and upon the eigenvectors [see Eq. (16)] and

$$\partial|c_i|/\partial d_j = \mathrm{sign}(c_i)\partial c_i/\partial d_j \tag{35}$$

where

$$\partial c_i/\partial d_j = -(\partial\mathbf{X}_i/\partial d_j)^T[M]\hat{e} - \mathbf{X}_i^T\{\partial[M]/\partial d_j\}\hat{e} \tag{36}$$

with all terms now considered as known quantities.

The derivative of the spectrum with respect to the design variable is

$$\partial\eta_m(\lambda_i)/\partial d_j = (\partial\eta_m/\partial\lambda_i)\partial\lambda_i/\partial d_j \tag{37}$$

Presumably $\partial\eta_m/\partial\lambda_i$ is known from the functional description of $\eta_m(\lambda)$ and, although there may be points of some spectra at which this derivative is undefined, this seems to pose no real problems.

The final term in Eq. (34) involves the derivative of the absolute value of the eigenvector which is a simple matter to determine given $\partial\mathbf{X}_i/\partial d_j$. Similarly the derivative may not be defined where a component vanishes but no actual difficulties seem to have arisen from this.

The derivatives of the upper bound stresses can be calculated in a similar manner to the displacements and we end this section with the conclusion that the partial derivatives of all behavior constraints applied to the problem can be computed with relatively low computational effort. To be sure there are some complex bookkeeping processes which must be executed but the computing time spent on these computations is small compared to the over-all time required.

7. Illustrative Examples

In all the examples presented in this section, a base shock in the form of one-half cycle sine pulse is used as the forcing function. This corresponds to

$$\ddot{\xi}_g(t) = A\,\sin(pt),\, 0 \leq t \leq \pi/p \tag{38}$$

where A is the magnitude of the shock (in./sec^2). The shock spectrum of a one-half cycle sine pulse representing the peak relative displacement is given in Chap. 4 of Ref. 6. An idealized form of this spectrum, shown in Fig. 1, is used in the present work. It is obtained by fitting piece-wise smooth curves to the actual spectrum. This not only facilitates the supply of shock spectral information to the computer, but also enables us to obtain directly the derivative of the response quantity with respect to the design variables.

A preliminary study using all eigenvectors ($r = n$) in the dynamic analysis was first made. The value of r for the approximate dynamic analysis used in each example is based on these results.

Example (1)

As a simple application, consider the planar truss-frame shown in Fig. 2 consisting of tubular members pinned together at the nodes. Members 1, 3, 4, and 5 are modeled with two general planar beam elements while the shortest member, number 2, is modeled with a single beam element. Thus, the structure has 26 degrees of freedom. The approximate dynamic analysis is performed by considering the first sixteen eigenvectors from the lowest end of the spectrum. The problem of optimum design consists of obtaining the

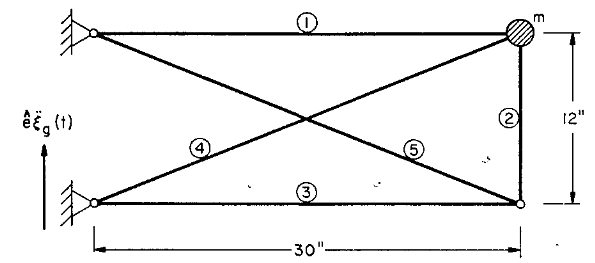

Fig. 2 Single bay truss-frame, example (1).

diameters d_1, d_2, d_3, d_4, and d_5 which give the lowest weight of the structure. The design variable δ_1 corresponds to members 1 and 3, δ_2 corresponds to member 2 and δ_3 corresponds to members 4 and 5. Thus, the problem has three design variables. A starting design with all members of mean diameter 1.6 in. weighs 19.23 lb. The limits on the behavior constraints and the side constraints are as follows:

$$y_{jm} \leq 0.2 \text{ in., for all } j, \; \sigma_{jm} \leq 45{,}000 \text{ psi, for all } j$$

$$25 \text{ Hz} \leq \omega_1 \leq 90 \text{ Hz}$$

$$140 \text{ Hz} \leq \omega_k, \; k = 2, 3, 4, \text{ and } 5$$

$$0.8 \text{ in.} \leq \delta_l \leq 2.5 \text{ in.}, \; l = 1, 2, 3$$

The proposed least weight design has a weight of 16.76 lb with

$$d_1 = d_3 = 1.5 \text{ in.}, \; d_2 = 0.86 \text{ in.}, \; d_4 = d_5 = 1.39 \text{ in.}$$

and five active constraints. The diameter of the vertical member, number 2 in Fig. 2, is at its lower bound. The second and third eigenvalues of the structure, λ_2 and λ_3, are at their lower bounds and the middle point of member, number 4, has a stress of 40.6 ksi/in.[2] All the feasible designs were observed to have the second eigenvalue λ_2 constraint active.

Example (2)

The two bay planar truss-frame of Fig. 3 consisting of tubular members pinned together at the nodes has 58 degrees of freedom if each member is modeled with two general planar beam elements. The system has 10 design variables as each member is allowed to change independently in the design space. The first fourteen eigenvectors (starting from the lowest end of the spectrum) were used to approximate the dynamic analysis. The limits on the behavior constraints and the side constraints were as follows:

$$y_{jm} \leq 0.2 \text{ in.}, \; \sigma_{jm} \leq 70{,}000 \text{ psi}$$

$$25 \text{ Hz} \leq \omega_1 \leq 90 \text{ Hz}$$

$$140 \text{ Hz} \leq \omega_k, \; k = 2, 3, 4, \text{ and } 5$$

$$0.8 \text{ in.} \leq \delta_l \leq 4.0 \text{ in.}, \; l = 1, 2, \ldots, 10$$

The first bounded design with member sizes d_1, d_2, d_4 = 2.15 in.; d_3 = 2.42 in.; d_5, d_6 = 2.49 in. and d_7, d_8, d_9, d_{10} = 1.91 in. has a weight of 29.85 lb and the second eigenvalue constraint λ_2 is bounded. The proposed least weight design has a weight of 19.03 lb with d_1 = 0.93 in., d_2 = 2.52 in., d_3 = 1.17 in., d_4 = 0.84 in., d_5 = 1.78 in., d_6 = 0.88 in., d_7 = 2.49 in., d_8 = 0.84 in., d_9 = 1.16 in., and d_{10} = 0.86 in. and has six active constraints. Members 4, 6, 8, and 10 in Fig. 3 are at the lower bounds and two eigenvalue constraints, λ_2 and λ_3 are at the lower bounds.

An interesting observation is that members 4, 6, and 10 are at the lower bounds and if we eliminate these members from the structure, the resulting structure is still statically stable. Hence, a modified structure eliminating these members from the start [see Fig. (4)] is considered. This produces a least weight design of 16.46 lb and is discussed in Ref. 2.

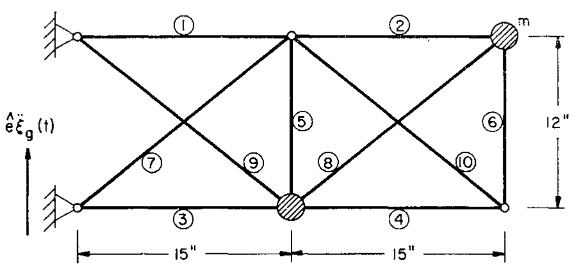

Fig. 3 Two bay truss-frame, examples (2) and (3).

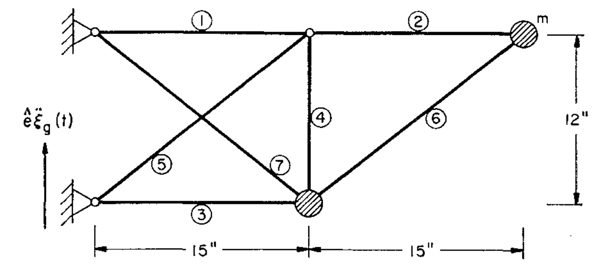

Fig. 4 Modified two bay truss-frame, example (2) modified.

Example (3)

The examples considered so far have eigenvalue constraints. In order to examine other possibilities, the structure of Fig. 4, is considered, this time with the reduced lower limits on the frequency constraints. Furthermore, the lower limit on the size of design variables and upper limit on maximum stress is also reduced. The new limitations imposed on the optimization problem were

$$y_{jm} \leq 0.2 \text{ in.}, \; \sigma_{jm} \leq 50{,}000 \text{ psi}$$

$$25 \text{ Hz} \leq \omega_1$$

$$0.6 \text{ in.} \leq \delta_l \leq 4.0 \text{ in.}, \; l = 1, \ldots, 7$$

A starting design with all members having a mean diameter of 2.5 in. and a weight of 25.21 lb happens to be bounded with the maximum stress at the middle point of member 3

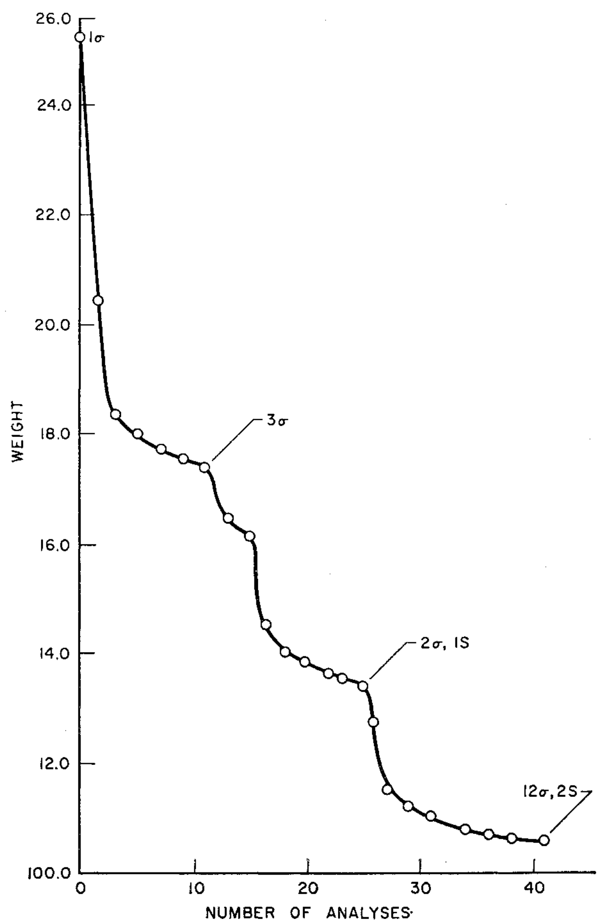

Fig. 5 Cumulative numer of analyses vs. weight for example (3).

of Fig. 4 at its upper bound. The proposed least weight design has a weight of 10.60 lb. Two members 4 and 5 of Fig. 4 are at the lower bounds in this design and members 1, 3, 6, and 7 are fully stressed since the maximum stresses at the middle and the end points of these members are at their upper bounds. Furthermore, the maximum stress at the middle and the end points of member 2 is almost at its upper bound. Thus, we observe that this particular proposed optimum design has either the members fully stressed or at their lower bounds. A plot of the cumulative number of analyses performed to obtain the bounded designs vs the weight of the bounded designs is shown in Fig. 5. Each open circle in Fig. 5 represents a design with active constraints. Some of the designs are indicated with the number and nature of the constraints which are active; for example 2σ, $1S$ means 2 stress constraints and 1 side constraint are active. As the design process progresses the number of constraints which are against their limit generally increases. The analysis utilizes 11 of the 41 possible eigenvectors for an approximation. The time (Univac 1108) taken for the analysis of the initial design was about 40 seconds while that for the intermediate analyses varied from 10 to 15 sec. Several other examples have been run and the results have been satisfactory.

Conclusions

In an effort of this type, which consists of the assembly of a capability and an examination of its feasibility, it is difficult to summarize the findings. The approach appears to be capable of solving the problem without an undue number of analyses per redesign and the computational time per analysis is reasonable. It is fair to say that the actual computer code used is not as efficient as it could be made and also there are some discretionary tolerances and convergence criteria in the program which could be liberalized to speed the solution.

The use of the minimization algorithm for the eigensolution seems to work quite well and the explicit partial derivatives of the response with respect to the design variables are accurate, easy to compute and their use greatly enhances the design process.

On the other hand, the class of structures examined is limited and attention should be given to whatever new problems may arise in an extention to the more realistic stiffened plate and shell type structures. In addition, should the shock spectral approach be inadequate for the analysis of dynamic behavior, more direct methods would have to be used and the result might be a loss in some of the explicit derivative information. It develops that Fourier analysis goes through in a form similar to the shock spectrum and would have similar properties; but other, more general methods pose some difficulty. Consideration could be given to stochastic dynamic response problems.

In conclusion, it is felt that computational approaches to structural optimization in the dynamic response regime are feasible and will form an important adjunct to modern design technology.

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